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PROPAGATION OF NONLINEAR WAVES IN A GAS–LIQUID MEDIUM. EXACT AND APPROXIMATE ANALYTICAL SOLUTIONS OF WAVE EQUATIONS

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A number of exact and approximate analytical solutions of the equations for one-dimensional and weakly non-one-dimensional waves propagating in a liquid with gas bubbles are presented for the case where the bubble distribution density is a continuous function of the bubble radius and spatial coordinates.

Key words: *bubble liquid, wave equation, exact solution.*

Introduction. The purpose of the study of the wave equations describing the propagation of linear and nonlinear waves in homogeneous and inhomogeneous multiphase media is to construct exact or approximate analytical solutions containing problem parameters in explicit form. Such solutions can be used to analyze and systematize experimental data and as tests in numerical calculations. References to papers dealing with analytical and numerical studies of wave processes in multiphase media are given in [1]. The present paper considers two previously unknown solutions for the case of one-dimensional waves propagating in a homogeneous gas–liquid medium. The first solution is an exact analytical solution of the Korteweg–de Vries–Burgers equation and has the form of a nonoscillating shock wave, and the second solution is an approximate self-similar solution of the axisymmetric Korteweg–de Vries equation in the form of a soliton with a long tail that ensures the satisfaction of the momentum conservation law. For non-one-dimensional linear equations, a number of exact solutions are obtained which describe the propagation of a bounded sound beam along the region of nonuniform bubble distribution in space. As is known, a bounded linear sound beam propagating in a homogeneous medium diverges [2]. The divergence angle is proportional to the ratio of the wavelength of the generated sound to the effective size of the source. In the present work, it is shown that the presence of inhomogeneity in the medium can lead to the existence of a solution in the form of nondivergent beams.

1. One-Dimensional Waves. The system of equations describing the propagation of one-dimensional nonlinear waves in an inhomogeneous gas–liquid medium [1] reduces to one equation (subscript 0 is omitted)

$$u_\tau + \alpha(\tau) u u_\eta + \beta(\tau) u_{\eta\eta\eta} - \mu(\tau) u_{\eta\eta} + \Big[\frac{k}{2\tau} + \delta(\tau)\Big] u = 0, \tag{1}$$

where the functions $\alpha(\tau)$, $\beta(\tau)$, $\mu(\tau)$, and $\delta(\tau)$ are determined in [1]; the values $k = 0$, 1, and 2 correspond to plane, cylindrical, and spherical waves, respectively. If $N_*(R_*, \tau) = N_*(R_*)$, i.e., the bubble size distribution density is identical at all points of space, the coefficients α, β, and μ are constants, $\delta = 0$, and Eq. (1) becomes the classical Korteweg–de Vries–Burgers equation

$$u_\tau + \alpha u u_\eta + \beta u_{\eta\eta\eta} - \mu u_{\eta\eta} + \frac{k}{2\tau}\, u = 0. \tag{2}$$

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Let us consider two solutions of Eq. (2).

1. In the case $k = 0$, transforming to a coordinate system moving at a certain constant velocity and integrating Eq. (2), we obtain a second-order ordinary differential equation. Numerical solutions of this equation obtained, for example, in [3, 4], are treated as weak shock waves with a monotonic or oscillating front. In the present work, we obtained an exact analytical solution of the form

$$u = u_*[2(1-\tanh\theta) + 1 - \tanh^2\theta] = u_*[2(1-\tanh\theta) + \cosh^{-2}\theta] = u_*[4(1-\tanh\theta) - (1-\tanh\theta)^2],$$
$$u_* = \frac{u_0}{2\alpha}, \quad \theta = m(\eta - u_0\tau), \quad u_0 = \frac{0.24\mu^2}{\beta}, \quad m = \frac{0.1\mu}{\beta}. \tag{3}$$

This solution is a nonoscillating shock wave which moves at constant velocity u_0. Solution (3) exists not for all values of m and u_0. For specified values of β and μ, the single values of the wavenumber m and the shock-wave amplitude u_0 are defined by formulas (3), and, vice versa, the ratio of β and μ is determined for a specified shock-wave amplitude; the wavenumber m is calculated from a specified amplitude and one of these parameters using (3). The question of the existence of an exact analytical solution for a shock wave with an oscillating front remains to be solved.

2. For $\mu = 0$ and $k = 1$, Eq. (2) is written as

$$u_\tau + \alpha u u_\eta + \beta u_{\eta\eta\eta} + \frac{1}{2\tau}\, u = 0. \tag{4}$$

The solution is obtained using the momentum and energy conservation laws, which in the adopted approximation for $u \to 0$ and $|\eta| \to \infty$ have the form

$$\tau^k \int\limits_{-\infty}^{\infty} u(\tau,\eta)\, d\eta = \tau_0^k \int\limits_{-\infty}^{\infty} u(\tau_0,\eta)\, d\eta = C_1 + O(\varepsilon); \tag{5}$$

$$\tau^k \int\limits_{-\infty}^{\infty} u^2(\tau,\eta)\, d\eta = \tau_0^k \int\limits_{-\infty}^{\infty} u^2(\tau_0,\eta)\, d\eta = C_2 + O(\varepsilon). \tag{6}$$

Here C_1 and C_2 are constants and $u(\tau_0, \eta)$ is the initial perturbation. The integration is performed from $-\infty$ because we consider wavelength of order $\varepsilon^{-1/2}$ at distances from the coordinate origin which exceed the characteristic perturbation width by a factor of ε^{-1} times. It should be noted that, because the momentum and energy conservation laws are one of the most important properties of the basic system of equations, it is natural to assume that, among all solutions of the approximate equation, only solutions that satisfy the conservation laws (5) and (6) have a physical meaning. It is easy to show that the energy conservation law is satisfied for any exact solution of Eq. (4) (for $u(\tau_0, \eta) \to 0$ and $|\eta| \to \infty$). The momentum conservation law is satisfied only for $C_1 = 0$:

$$\int\limits_{-\infty}^{\infty} u(\tau,\eta)\, d\eta = \int\limits_{-\infty}^{\infty} u(\tau_0,\eta)\, d\eta = 0. \tag{7}$$

The vanishing of the integral of the initial perturbation $u(\tau_0, \eta)$ is a necessary and sufficient condition for the satisfaction of the momentum conservation law for all $\tau > \tau_0$. Hence, first, a cylindrical wave propagating in one direction has both compression phases and rarefaction phases; second, in the axisymmetric case, Eq. (4) is applicable beginning from the moment when any initial perturbation evolves into a wave that satisfies condition (7). Before this moment, the waves propagating in both directions play a significant role in the evolution of the initial perturbation, and Eq. (4) is not sufficient for the description of this evolution. For large values of τ, an asymptotic solution of Eq. (4) can be sought as a series in the inverse powers of τ:

$$u(\tau,\eta) = \tau^{-1/2} \sum_{m=0}^{\infty} u_m \tau^{-m}. \tag{8}$$

Substitution of (8) into Eq. (4) yields the linear Korteweg–de Vries equation for u_0. The asymptotic solution of this equation is known and is therefore not given in the present paper. The subsequent approximations are obtained in the form of a recursive system of linear equations. Thus, for large values of τ, divergent cylindrical waves are a spreading wave train with decay of order $\tau^{-5/6}$ (because in the asymptotic solution, decay follows the law $\tau^{-1/3}$).

However, it is possible to specify initial conditions under which cylindrical waves of the type of solitons exist for large values of τ.

Let the initial perturbation have the form

$$u(\tau_0,\eta)=\lambda^{-2/3}f(\lambda^{-1/3}\eta),\qquad \int\limits_{-\infty}^{\infty} f(\lambda^{-1/3}\eta)\,d\eta=0, \tag{9}$$

where the small parameter λ satisfies the relation $\varepsilon \ll \lambda \ll 1$. Substitution of (9) into (6) yields

$$\lambda\tau\int\limits_{-\infty}^{\infty} u^2(\tau,\eta)\,d\eta=\tau_0\int\limits_{-\infty}^{\infty} f^2(z)\,dz. \tag{10}$$

Making a changing of the function and independent variables, we distinguish the main part of Eq. (4) in the form of the Korteweg–de Vries equation for a plane wave, and introduce the small parameter λ in explicit form into this equation:

$$u(\tau,\eta)=\frac{16\beta}{9\alpha(\lambda\tau)^{2/3}}\,U(t,r); \tag{11}$$

$$t=\frac{64\beta}{27\lambda}\ln\frac{\tau}{\tau_0},\qquad r=\frac{4\eta}{3(\lambda\tau)^{1/3}}. \tag{12}$$

Equations (9)–(12) lead to the relations

$$\int\limits_{-\infty}^{\infty} U^2(t,r)\,dr=C,\qquad \int\limits_{-\infty}^{\infty} U(t,r)\,dr=0, \tag{13}$$

where C is a constant. Substitution of (11) and (12) into (4) yields the equation for $U(t,r)$

$$\frac{1}{(\lambda\tau)^{5/3}}\left[U_t+UU_r+U_{rrr}-\lambda_*(U+2rU_r)\right]=0, \tag{14}$$

where $\lambda_*=9\lambda/(32\beta)$ is a new small parameter. The function $U(t,r)$ can be represented as a series in the powers of λ:

$$U(t,r)=\sum_{m=0}^{\infty}U_m(t,r)\lambda_*^m. \tag{15}$$

Substitution of (15) into (14) yields the Korteweg–de Vries equation for the function $U_0(t,r)$

$$U_{0t}+U_0U_{0r}+U_{0rrr}=0. \tag{16}$$

In this case, for the function U_0 from (13), it follows that the energy and momentum conservation laws have the form

$$\int\limits_{-\infty}^{\infty} U_0^2(t,r)\,dr=C,\qquad \int\limits_{-\infty}^{\infty} U_0(t,r)\,dr=0. \tag{17}$$

The solution of Eq. (16) with conditions (17) is well-known. In particular, it is known that, depending on the value of the parameter σ, which, at the initial time, is equal to the ratio of the nonlinear term to the dispersion term, Eq. (16) can have two solutions. For values of σ smaller than a certain critical value σ_0, the solution is a spreading wave train, which is similar in the rate of decrease in the amplitude and phase motion to the solution of the linearized Korteweg–de Vries equation. For $\sigma>\sigma_0$ and $\tau\to\infty$, the solution is represented in the form of one or several solitons:

$$U_{0i}(t,r)=a_i\cosh^{-2}\left[\sqrt{a_i/12}\,(r-a_it/3)\right]. \tag{18}$$

However, in order that a solution of the type (18) satisfy Eq. (14), it is necessary that the terms of this equation omitted in the zero approximation be small for $t \to \infty$. This condition is satisfied if solution (18) is represented as

$$U_i(x) = 12k_i^2 \cosh^{-2} X, \tag{19}$$

where $X = k_i Z$, $k_i = \sqrt{a_i/12}$, and $Z = r - \psi_i(t)$.

The functions $\psi_i(t)$ are determined from the condition of stationarity of Eq. (14) in the coordinates t and Z, i.e., this equation should have the form

$$-4k^2 U_{iX} + U_i U_{iX} + k^2 U_{iXXX} - \lambda_*(U_i + 2XU_{iX}) = 0. \tag{20}$$

If the above condition is satisfied, the functions $\psi_i(t)$ can be expressed as

$$\psi_i(t) = 4k_i^2 + (r_{i*} - 4k_i^2)\exp\,[-\lambda_*(t - t_{i*})/2]. \tag{21}$$

Here $a_i = 12k_i^2$ is the amplitude of the ith soliton, t_{i*} is the time the ith soliton appears and separates from the oscillating tail, and r_{i*} is the coordinate of the ith soliton. With relation (12) and the parameter $\lambda_* = 9\lambda/(32\beta)$ taken into account, formula (21) can be written as

$$\psi_i(t) = 4k_i^2 + (r_* - 4k_i^2)(\tau/\tau_{i0})^{-1/3}. \tag{22}$$

From (19)–(22), it follows that the terms of the equation omitted in the zero approximation (14) remain small quantities of order λ_* for any $t \to \infty$. Thus, under certain conditions, cylindrical waves can exist in the form of solitons. The time and place of occurrence of solitons are determined only in numerical calculations, but because these constants are not included in formulas (11) and (12), the laws governing the amplitude decay and soliton extension are completely defined. From formula (11), it follows that the amplitudes of the solitons decay more slowly (decay of order $\tau^{-2/3}$) than the amplitude of the linear wave train (decay of order $\tau^{-5/6}$).

Returning to Eq. (1), we note that, in the case of variable coefficients, the role of the conservation laws in seeking solutions of this equation is not less important than in the problem considered above. For $\mu = 0$ and $u(\tau,\eta) \to 0$ as $|\eta| \to \infty$ in the case of variable coefficients, the momentum and energy conservation laws are written as

$$[1 - \varphi_*(\tau)]\tau^k c(\tau) \int\limits_{-\infty}^{\infty} u(\tau,\eta)\,d\eta = C_1 + O(\varepsilon); \tag{23}$$

$$[1 - \varphi_*(\tau)]\tau^k c(\tau) \int\limits_{-\infty}^{\infty} u^2(\tau,\eta)\,d\eta = C_2 + O(\varepsilon), \tag{24}$$

where $\varphi_*(\tau)$ is the unperturbed volumetric concentration of the gas.

It is easy to show that, for $\mu = 0$, the energy conservation law (24) is satisfied for any exact solution of Eq. (1). If $\delta(\tau) \neq 0$, the momentum conservation law (23) is satisfied only for $C_1 = 0$ even for $k = 0$, i.e., for plane waves, which, unlike in the case of cylindrical waves, does not satisfy the existence condition for solitary waves. In this case, one should apparently use the equations of the next, first, approximation which admit the occurrence of waves whose length is an order of magnitude greater than the wavelength in the zero approximation. Inclusion of such waves in the solution without violation (because of smallness of their amplitudes) of the square-law energy conservation law can lead to satisfaction of the linear momentum conservation law (23). Physically, the occurrence of waves with length comparable to the characteristic size of bubble distribution nonuniformity in the liquid is natural. However, this question requires a separate investigation.

To study analytical solutions of Eq. (1), we make a change of the required function and independent variables:

$$u(\tau,\eta) = f(\tau)U[X, T(\tau)], \qquad X = q(\tau)\eta. \tag{25}$$

As for Eq. (4), the functions $f(\tau)$, $q(\tau)$, and $T(\tau)$ are determined from the conditions of separation of the main part of Eq. (1) in the form of the Korteweg–de Vries equation and the conditions of existence of conservation laws in the form of

$$\int_{-\infty}^{\infty} U^2(T,X)\,dX = C, \qquad \int_{-\infty}^{\infty} U(T,X)\,dX = 0. \tag{26}$$

Equations (25) and (26) lead to

$$q(\tau) = \alpha^2(\tau)[1-\varphi_*(\tau)]^{-1}\tau^{-k}\beta^{-2}(\tau)^{1/3}c^{-1/2}(\tau), \qquad f(\tau) = \beta(\tau)\alpha^{-1}(\tau)q^2(\tau),$$
$$T(\tau) = \int_{\tau_0}^{\tau} \alpha^2(z)[1-\varphi_*(z)]^{-1}z^{-k}\beta^{-1}(z)c^{-1}(z)\,dz. \tag{27}$$

In this case, the equation for $U(T,X)$ becomes

$$U_T + UU_X + U_{XXX} - F(\tau)\Big[\frac{k}{\tau} + Q(\tau)\Big](U + 2XU_X) = 0, \tag{28}$$

where

$$F(\tau) = \frac{1}{6\beta(\tau)q^3(\tau)}, \qquad Q(\tau) = \frac{\partial}{\partial\tau}\Big\{\ln\frac{[1-\varphi_*(\tau)]\beta^2(\tau)c(\tau)}{\alpha^2(\tau)}\Big\}. \tag{29}$$

Because $k/\tau = \partial\ln(\tau^k)/\partial\tau$, Eq. (28) can be written as

$$U_T + UU_X + U_{XXX} - P(\tau)(U + 2XU_X) = 0, \tag{30}$$

where

$$P(\tau) = \frac{1}{6\beta(\tau)q^3(\tau)}\,\frac{\partial}{\partial\tau}\Big\{\ln\frac{\tau^k[1-\varphi_*(\tau)]\beta^2(\tau)c(\tau)}{\alpha^2(\tau)}\Big\}. \tag{31}$$

For $P(\tau) = 0$, Eq. (30) becomes the classical Korteweg–de Vries equation with the conservation laws (26). From formula (31), it follows that, to satisfy this condition, it is sufficient that the expression in braces should be a constant. Thus, we obtain the following equation for the bubble radius distribution density:

$$\frac{\tau^k[1-\varphi_*(\tau)]\beta^2(\tau)c(\tau)}{\alpha^2(\tau)} = C, \tag{32}$$

where C is a constant. Comparing (32) and (27), we obtain

$$q = C^{-3}, \qquad f = \frac{\beta(\tau)}{\alpha(\tau)}C^{-3}, \qquad T = C^{-3}\int \beta(z)\,dz.$$

Under the assumption $P(\tau) = \lambda$ (λ is a small parameter), the problem reduces to the problem considered above for cylindrical waves. In this case, all solutions obtained previously are also valid for Eq. (30), but the variation in soliton characteristics such as the amplitude, width, and velocity of motion is determined from (27).

2. Weakly Non-One-Dimensional Waves. The system of equations describing the propagation of nonlinear non-one-dimensional waves can be obtained from formulas (53) and (54) of [1] for $k = 0$ (subscript 0 is omitted):

$$u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \delta(\tau)u - c_1(\tau,y,z)u_\eta + (v_y + w_z)/2 = 0,$$
$$v_\eta = c(\tau)u_y, \qquad w_\eta = c(\tau)u_z. \tag{33}$$

Eliminating the functions v and w from (33), we obtain the following equation for u:

$$[u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \delta(\tau)u - c_1(\tau,y,z)u_\eta]_\eta + c(\tau)(u_{yy} + u_{zz})/2 = 0. \tag{34}$$

An equation similar to (34) was first derived in [5] for surface waves in a reservoir of finite depth [$c(\tau) = 1$ and $\mu(\tau) = \delta(\tau) = 0$; α and β are constant]. Later this equation (for $c_1 = 0$) became known as the Kadomtsev–Petviashvili equation. If we eliminate the dispersion term from Eq. (34) and add the diffusion term to this equation, it becomes the well-known Khokhlov–Zabolotskaya equation derived for the propagation of nonlinear sound beams in a viscous liquid. It should be noted that, for a bounded nonlinear beam, as for the Kadomtsev–Petviashvili equation, no exact solution has been obtained. The propagation of a signal generated by an axisymmetric source is described by the equation

$$[u_\tau + \alpha(\tau)uu_\eta + \beta(\tau)u_{\eta\eta\eta} - \mu(\tau)u_{\eta\eta} + \delta(\tau)u - c_1(\tau, r)u_\eta]_\eta + \frac{c(\tau)}{2}\Big(u_{rr} + \frac{1}{r}\,u_r\Big) = 0. \tag{35}$$

Below, we consider linear equations which describe the propagation of a sound beam generated by a periodic source of finite width in a medium with constants α, β, and μ and, hence, with $\delta = 0$ for $c_1(\tau, y) \neq 0$. For a plane source, Eq. (35) is written as

$$u_{\tau\eta} + \beta u_{\eta\eta\eta\eta} - \mu u_{\eta\eta\eta} - c_1(y)u_{\eta\eta} + u_{yy} = 0. \tag{36}$$

Setting $\mu = 0$, we seek a solution of Eq. (36) in the form

$$u(x, y) = a(by)\cos(kx), \qquad x = \eta + V\tau. \tag{37}$$

Substitution of (37) into (36) yields

$$[-k^2V + k^4\beta + k^2c_1(by)a(by) + b^2a''(by)]\cos(kx) = 0.$$

For the specified function $c_1(by)$, the condition of vanishing of the expression in square brackets leads to the following linear equation for the function $a(by)$:

$$b^2a'' + (k^4\beta + k^2c_1(by) - k^2V)a = 0. \tag{38}$$

Let $c_1(by) = c_0\cosh^{-2}(by)$. Equation (38) leads to

$$a(by) = a_0\cosh^{-2}(by), \qquad V = k^2\beta + 2c_0/3, \qquad b = k(c_0/6)^{1/2}; \tag{39}$$

therefore,

$$u(x, y) = a_0\cosh^{-2}(by)\cos(kx). \tag{40}$$

Function (40) represents a bounded monochromatic sound beam which is symmetric about y. Because Eq. (38) is linear, its solution can be represented as a superposition of harmonics of different frequencies. In the present paper, one harmonic is considered. From Eq. (38), the function $c_1(by)$ can be expressed as

$$c_1(by) = V - k^2\beta - \frac{b^2a''(by)}{k^2a(by)}. \tag{41}$$

Specifying the dependence of the oscillation amplitude in the beam on y, from formula (41) we determine the nonuniformity of the bubble distribution in the liquid that provides the existence of the beam considered. For example, for $a(by) = a_0\cosh^{-m}(by)$, we have

$$c_1(by) = V - k^2\beta - m^2b^2k^{-2} + m(m+1)b^2k^{-2}\cosh^{-2}(by).$$

The condition of vanishing of c_1 at infinity leads to $V = k^2\beta + m^2b^2k^{-2}$.

In the coordinates $x = \eta + V\tau$ and r, the linear equation for the propagation of an axisymmetric sound beam is written as

$$Vu_{xx} + \beta u_{xxxx} - \mu u_{xxx} - c_1(r)u_{xx} + u_{rr} + \frac{1}{r}\,u_r = 0. \tag{42}$$

We seek a solution of Eq. (42) ($\mu = 0$) in the form

$$u(x, r) = a(r)\cos(kx). \tag{43}$$

In this case, formula (41) is written as

$$c_1(r) = V - k^2\beta - \frac{a''(r)}{k^2a(r)} - \frac{a'(r)}{k^2ra(r)}.$$

Specifying $a(r)$ in the form $a(r) = a_0\exp[-(1 + b^2r^2)^{1/2}]$, we obtain

$$c_1(r) = b^2k^{-2}[(1 + b^2r^2)^{-1/2} + (1 + b^2r^2)^{-1} + (1 + b^2r^2)^{-3/2}]. \tag{44}$$

The parameter $V = k^2\beta + b^2k^{-2}$ is determined from the condition of vanishing of c_1 at infinity. From formula (44), it follows that, as $r \to \infty$, the function $c_1(r)$ decreases monotonically to zero but much more slowly than the amplitude of the generated signal.

Solutions of Eq. (36) with nonzero viscosity are given by formulas (40) and (43) whose right sides are supplemented by the multiplier $\exp(-k^2\mu\tau)$.

Thus, it is shown that a nonuniform bubble distribution on the coordinates can lead to a solution in the form of nonexpanding sound beams, unlike in the case a homogeneous gas–liquid medium. The method of derivation of approximate wave equations proposed in [1] and the solutions of these equations given in the present work can be used to study the effect of bounded sound beams on the bubble distribution in a liquid. In this case, the basic system of equations should be written taking into account the relative motion of bubbles and liquid.

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